



## On the Irreducibility of Perron Representations of Degrees 4 and 5

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**Abstract.** We consider the graph  $E_{n+1,1}$  with  $(n+1)$  generators  $\sigma_1, \dots, \sigma_n$ , and  $\delta$ , where  $\sigma_i$  has an edge with  $\sigma_{i+1}$  for  $i = 1, \dots, n+1$ , and  $\sigma_1$  has an edge with  $\delta$ . We then define the Artin group of the graph  $E_{n+1,1}$  for  $n = 3$  and  $n = 4$  and consider its reduced Perron's representation of degrees four and five respectively. After we specialize the indeterminates used in defining the representation to non-zero complex numbers, we obtain necessary and sufficient conditions that guarantee the irreducibility of the representations for  $n = 3$  and  $4$ .

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### 1. Introduction

Let  $\Gamma$  be an undirected simple graph. The Artin group  $A$  is defined as an abstract group whose generators are the vertices of  $\Gamma$  that satisfy the two relations:  $xy = yx$  for vertices  $x$  and  $y$  that have no edge in common and  $xyx = yxy$  if the vertices  $x$  and  $y$  have a common edge.

Having defined  $A$ , we consider the graph  $A_n$  having  $n$  vertices  $\sigma_i$ 's ( $1 \leq i \leq n$ ) in which  $\sigma_i$  and  $\sigma_{i+1}$  share a common edge, where  $i = 1, 2, \dots, n-1$ . Indeed, the Artin group of  $A_n$ , denoted by  $A(A_n)$ , is the braid group on  $n+1$  strands,  $B_{n+1}$ . That is,  $A(A_n) = B_{n+1}$ .

From the graph  $A_n$ , we obtain the graph  $E_{n+1,p}$  by adding a vertex  $\delta$  and an edge connecting  $\sigma_p$  and  $\delta$ . Here  $1 \leq p \leq n$ . Clearly, the graph  $A_n$  embeds in the graph  $E_{n+1,p}$ . Consequently,  $A(A_n) \subset A(E_{n+1,p})$ . As a result, a representation of  $A(E_{n+1,p})$  yields a representation of  $B_{n+1}$ .

Perron's strategy is to begin with the reduced Burau representation of  $B_{n+1}$  of degree  $n$  and extend it to a representation of  $B_{n+1}$  of degree  $2n$ . The representation obtained is referred to as Burau bis representation. Next, Perron constructs for each  $\lambda = (\lambda_1, \dots, \lambda_n)$

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a representation  $\psi_\lambda : A(E_{n+1,p}) \rightarrow GL_{2n}(Q(t, d_1, \dots, d_n))$ , where  $t, d_1, \dots, d_n, \lambda_1, \dots, \lambda_n$  are indeterminates.

In [3], we determined necessary and sufficient condition that guarantees the irreducibility of the representation  $\psi_\lambda$  for  $n = 2$ . In our work, we extend our work to  $n = 3$  and  $n = 4$ . We reduce the complex specialization of the representation  $\psi_\lambda$  to representations of  $A(E_{4,1})$  and  $A(E_{5,1})$  of degrees 4 and 5 respectively. In each case, a necessary and sufficient condition which guarantees the irreducibility of the considered representation is obtained. The obtained conditions are similar to the condition obtained in the case  $n = 2$ , which was studied in [3].

## 2. Burau bis Representation

The Burau Bis representation is a representation of  $B_{n+1}$  of degree  $2n$ . It is defined as follows:

$$\psi : B_{n+1} \rightarrow GL_{2n}(\mathbb{Z}[t, t^{-1}])$$

$$\psi(\sigma_i) = \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix}, \quad 1 \leq i \leq n$$

Here,  $R_i$  denotes an  $n \times n$  block of zeros with a  $t$  placed in the  $(i, i)$ th position and  $I_n$  denotes the  $n \times n$  identity matrix.

This representation was constructed by Perron by extending the reduced Burau representation of degree  $n$  to a representation of  $B_{n+1}$  of degree  $2n$ .

The reduced Burau representation  $B_{n+1} \rightarrow GL_n(\mathbb{Z}[t, t^{-1}])$  is defined as follows:

$$\sigma_i \rightarrow J_i = \left( \begin{array}{c|cc|c} I_{i-2} & & 0 & 0 \\ \hline & 1 & 0 & 0 \\ 0 & t & -t & 1 \\ & 0 & 0 & 1 \\ \hline 0 & & 0 & I_{n-i-1} \end{array} \right),$$

where  $I_k$  stands for the  $k \times k$  identity matrix. Here,  $i = 2, \dots, n-1$ .

$$\sigma_1 \rightarrow J_1 = \left( \begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & \\ \hline 0 & & I_{n-2} \end{array} \right)$$

$$\sigma_n \rightarrow J_n = \left( \begin{array}{c|cc} I_{n-2} & & 0 \\ \hline 0 & 1 & 0 \\ & t & -t \end{array} \right)$$

For more details, see [2] and [5].

### 3. Perron Representation

The Burau bis representation extends to  $A(E_{n+1,p})$  for all possible values of  $n$  and  $p$  in the following way.

We define the following  $n \times n$  matrices:

$$A = (\lambda_1 b, \lambda_2 b, \dots, \lambda_n b)$$

$$B = (0, \dots, 0, b, 0, \dots, 0)$$

$$C = (\lambda_1 d, \lambda_2 d, \dots, \lambda_n d)$$

$$D = (0, \dots, 0, d, 0, \dots, 0),$$

where 0 denotes a column of  $n$  zeros,  $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ ,  $d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ , and  $\lambda = (\lambda_1, \dots, \lambda_n)$ .

For each  $i = 1, \dots, n$ , we have that  $b_i$  satisfies the following conditions

$$tb_i = -td_{i-1} + (1+t)d_i - d_{i+1}, \quad i \neq p,$$

$$tb_p = -td_{p-1} + (1+t)d_p - d_{p+1} + t,$$

$$\sum_{i=1}^n \lambda_i b_i = -(1 + d_p + t),$$

setting any undefined  $d_j$  equal zero.

For any choice  $\lambda = (\lambda_1, \dots, \lambda_n)$ , we get a linear representation

$$\psi_\lambda : A(E_{n+1,p}) \rightarrow Gl_{2n}(R),$$

where  $R$  is the field of rational fractions in  $n+1$  indeterminates  $\mathbb{Q}(t, d_1, \dots, d_n)$ .

$$\psi_\lambda(\sigma_i) \rightarrow \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix},$$

$$\psi_\lambda(\delta) \rightarrow \begin{pmatrix} I_n + A & B \\ C & I_n + D \end{pmatrix}.$$

For more details, see [2].

#### 4. Reducibility of $\psi_\lambda : A(E_{4,1}) \rightarrow GL_6(\mathbb{C})$

Having defined Perron's representation, we set  $n = 3$  and  $p = 1$  to get the following vectors.  $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ ,  $d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ , and  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ .

After we specialize the indeterminate  $d_3$  to  $\frac{-t(1+t+t^2)}{1+t}$ , we get the following  $3 \times 3$  matrices:

$$A = \begin{pmatrix} \lambda_1 b_1 & \lambda_2 b_1 & \lambda_3 b_1 \\ \lambda_1 b_2 & \lambda_2 b_2 & \lambda_3 b_2 \\ \lambda_1 b_3 & \lambda_2 b_3 & \lambda_3 b_3 \end{pmatrix},$$

$$B = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \lambda_1 d_1 & \lambda_2 d_1 & \lambda_3 d_1 \\ \lambda_1 d_2 & \lambda_2 d_2 & \lambda_3 d_2 \\ \frac{-t(1+t+t^2)}{1+t} \lambda_1 & \frac{-t(1+t+t^2)}{1+t} \lambda_2 & \frac{-t(1+t+t^2)}{1+t} \lambda_3 \end{pmatrix},$$

and

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ d_2 & 0 & 0 \\ \frac{-t(1+t+t^2)}{1+t} & 0 & 0 \end{pmatrix}.$$

Simple computations show that the parameters satisfy the following equations:

- $tb_2 = -td_1 + (1+t)d_2 + \frac{t(1+t+t^2)}{1+t}$
- $tb_3 = -td_2 - t(1+t+t^2)$
- $tb_1 = (1+t)d_1 - d_2 + t$
- $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = -(1+t+d_1)$

Having defined the  $3 \times 3$  matrices  $A$ ,  $B$ ,  $C$  and  $D$ , we obtain the multiparameter representation  $A(E_{4,1})$ . This representation is of degree 6. We specialize the parameters  $\lambda_1, \lambda_2, \lambda_3, b_1, b_2, b_3, d_1, d_2, t$  to values in  $\mathbb{C} - \{0\}$ . We further assume that  $t \neq -1$ . The representation  $\psi_\lambda : A(E_{4,1}) \rightarrow GL_6(\mathbb{C})$  is defined as follows:

$$\psi_{\lambda}(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ t & 0 & 0 & -t & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_{\lambda}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & t & 0 & t & -t & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_{\lambda}(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 0 & t & -t \end{pmatrix},$$

and

$$\psi_{\lambda}(\delta) = \begin{pmatrix} 1 + \lambda_1 b_1 & \lambda_2 b_1 & \lambda_3 b_1 & b_1 & 0 & 0 \\ \lambda_1 b_2 & 1 + \lambda_2 b_2 & \lambda_3 b_2 & b_2 & 0 & 0 \\ \lambda_1 b_3 & \lambda_2 b_3 & 1 + \lambda_3 b_3 & b_3 & 0 & 0 \\ \lambda_1 d_1 & \lambda_2 d_1 & \lambda_3 d_1 & 1 + d_1 & 0 & 0 \\ \lambda_1 d_2 & \lambda_2 d_2 & \lambda_3 d_2 & d_2 & 1 & 0 \\ \frac{-t(1+t+t^2)}{1+t} \lambda_1 & \frac{-t(1+t+t^2)}{1+t} \lambda_2 & \frac{-t(1+t+t^2)}{1+t} \lambda_3 & \frac{-t(1+t+t^2)}{1+t} & 0 & 1 \end{pmatrix}.$$

The graph  $E_{4,1}$  has 4 vertices  $\sigma_1, \sigma_2, \sigma_3$  and  $\delta$ . Since  $p = 1$ , it follows that the vertex  $\delta$  has a common edge with  $\sigma_p = \sigma_1$ . Therefore, the following relations are satisfied.

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \quad (4.1)$$

$$\sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3 \quad (4.2)$$

$$\sigma_1 \sigma_3 = \sigma_3 \sigma_1 \quad (4.3)$$

$$\sigma_2 \delta = \delta \sigma_2 \quad (4.4)$$

$$\sigma_3 \delta = \delta \sigma_3 \quad (4.5)$$

$$\sigma_1 \delta \sigma_1 = \delta \sigma_1 \delta \quad (4.6)$$

We note that relations (4.1), (4.2), and (4.3) are actually Artin's braid relation of the classical braid group,  $B_4$  having  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  as standard generators. This assures that a representation of  $A(E_{4,1})$  yields a representation of  $B_4$ .

For more details, see [1] and [4].

**Lemma 1.** *The representation  $\psi_\lambda : A(E_{4,1}) \rightarrow GL_6(\mathbb{C})$  is reducible.*

*Proof.* For simplicity, we write  $\sigma_i$  instead of  $\psi_\lambda(\sigma_i)$ . The subspace  $S = \left\langle e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3, e_4, e_5, e_6 \right\rangle$  is an invariant subspace of dimension 4. To see this:

- (i)  $\sigma_1(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) = e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + te_4 \in S$
- (ii)  $\sigma_2(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) = e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + t\frac{b_2}{b_1}e_5 \in S$
- (iii)  $\sigma_3(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) = e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + t\frac{b_3}{b_1}e_6 \in S$
- (iv)  $\delta(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) = (1 + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) + \frac{d_1}{b_1}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)e_4 + \frac{d_2}{b_1}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)e_5 + \frac{-(1+t+t^2)}{b_1(1+t)}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)e_6 \in S$
- (v)  $\sigma_1 e_4 = -te_4 \in S$
- (vi)  $\sigma_2 e_4 = e_4 + te_5 \in S$
- (vii)  $\sigma_3 e_4 = e_4 \in S$
- (viii)  $\delta e_4 = b_1(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) + (1 + d_1)e_4 + d_2 e_5 \frac{-t(1+t+t^2)}{1+t} e_6 \in S$
- (ix)  $\sigma_1 e_5 = e_4 + e_5 \in S$
- (x)  $\sigma_2 e_5 = -te_5 \in S$
- (xi)  $\sigma_3 e_5 = e_5 + te_6 \in S$
- (xii)  $\delta e_5 = e_5 \in S$
- (xiii)  $\sigma_1 e_6 = e_6 \in S$
- (xiv)  $\sigma_2 e_6 = e_5 + e_6 \in S$
- (xv)  $\sigma_3 e_6 = -te_6 \in S$

$$(xvi) \quad \delta e_6 = e_6 \in S$$

### 5. On the Irreducibility of $\psi'_\lambda : A(E_{4,1}) \rightarrow GL_4(\mathbb{C})$

We consider the representation  $\psi_\lambda : A(E_{4,1}) \rightarrow GL_6(\mathbb{C})$  restricted to the basis  $e_1, e_2, e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3, e_4, e_5$ , and  $e_6$ . The matrix of  $\sigma_1$  becomes

$$\psi_\lambda(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We reduce our representation to a 4-dimensional one by considering the sub-basis  $e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3, e_4, e_5$ , and  $e_6$  to get  $\psi'_\lambda : A(E_{4,1}) \rightarrow GL_4(\mathbb{C})$ . The representation is defined as follows:

$$\psi'_\lambda(\sigma_1) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & -t & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_2) = \begin{pmatrix} 1 & 0 & \frac{tb_2}{b_1} & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & \frac{tb_3}{b_1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & -t \end{pmatrix},$$

and

$$\psi'_\lambda(\delta) =$$

$$\begin{pmatrix} 1 + \sum_{i=1}^3 \lambda_i b_i & \frac{d_1}{b_1}(\sum_{i=1}^3 \lambda_i b_i) & \frac{d_2}{b_1}(\sum_{i=1}^3 \lambda_i b_i) & \frac{-t(1+t+t^2)}{b_1(1+t)}(\sum_{i=1}^3 \lambda_i b_i) \\ b_1 & 1 + d_1 & d_2 & -t(1+t+t^2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We then diagonalize the matrix corresponding to  $\psi'_\lambda(\sigma_1)$  by an invertible matrix, say  $T$ , and conjugate the matrices of  $\psi'_\lambda(\sigma_2)$ ,  $\psi'_\lambda(\sigma_3)$ , and  $\psi'_\lambda(\delta)$  by the same matrix  $T$ . The invertible matrix  $T$  is given by

$$T = \begin{pmatrix} 0 & 0 & 1 & t \\ 0 & 0 & 0 & -1-t \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In fact, a computation shows that

$$T^{-1}\psi'_\lambda(\sigma_1)T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}.$$

After conjugation, we get

$$T^{-1}\psi'_\lambda(\sigma_2)T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{-t^2}{1+t} & 0 & \frac{-(1+t+t^2)}{1+t} \\ 0 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} & 1 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ 0 & \frac{-t}{1+t} & 0 & \frac{1}{1+t} \end{pmatrix},$$

$$T^{-1}\psi'_\lambda(\sigma_3)T = \begin{pmatrix} -t & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ \frac{tb_3}{b_1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$T^{-1}\psi'_\lambda(\delta)T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-t(1+t+t^2)}{(1+t)^2} & 1 + \frac{d_2}{1+t} & \frac{b_1}{1+t} & \frac{t}{1+t} \\ \frac{-t(1+t+t^2)}{b_1(1+t)}k & \frac{d_2}{b_1}k & 1+k & \frac{(-d_1(1+t)+d_2+b_1t)((\sum_{i=1}^3 \lambda_i b_i)(1+t)+b_1t)}{b_1(1+t)} \\ \frac{t(1+t+t^2)}{1+t} & \frac{-d_2}{1+t} & \frac{-b_1}{1+t} & \frac{1}{1+t} \end{pmatrix}.$$

where  $k = \frac{b_1 t}{1+t} + \sum_{i=1}^3 \lambda_i b_i$ .

The entries of the matrices  $T^{-1}\psi'_\lambda(\sigma_2)T$  and  $T^{-1}\psi'_\lambda(\delta)T$  are well-defined since we assume in our work that  $t \neq -1$ . For simplicity, we denote  $T^{-1}\psi'_\lambda(\sigma_1)T$  by  $\psi'_\lambda(\sigma_1)$ ,  $T^{-1}\psi'_\lambda(\sigma_2)T$  by  $\psi'_\lambda(\sigma_2)$ ,  $T^{-1}\psi'_\lambda(\sigma_3)T$  by  $\psi'_\lambda(\sigma_3)$ , and  $T^{-1}\psi'_\lambda(\delta)T$  by  $\psi'_\lambda(\delta)$ .

We now prove some lemmas and propositions to determine a sufficient and necessary condition for irreducibility of  $\psi'_\lambda : A(E_{4,1}) \rightarrow GL_4(\mathbb{C})$ .

**Lemma 2.** *The proper subspace  $S = \langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \rangle$  is not invariant if and only if  $t^4 + t^3 + t^2 + t + 1 \neq 0$ .*

*Proof.* First, we prove that proper subspace  $S = \langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \rangle$  is not invariant if  $t^4 + t^3 + t^2 + t + 1 \neq 0$ .

Assume, for contradiction, that  $S$  is invariant.

$$\text{We have } \psi'_\lambda(\sigma_2)(e_4) = \begin{pmatrix} 1 \\ \frac{-(1+t+t^2)}{1+t} \\ \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} \in S.$$

This implies that  $(1+t+t^2)b_3 = -t(b_2+tb_2+tb_1)$ .

By using the equations:  $tb_2 = -td_1 + (1+t)d_2 + \frac{t(1+t+t^2)}{1+t}$ ,  $tb_3 = -td_2 - t(1+t+t^2)$ , and  $tb_1 = (1+t)d_1 - d_2 + t$ , simple computations give  $t^4 + t^3 + t^2 + t + 1 = 0$ , a contradiction.

On the other hand, we assume that  $t^4 + t^3 + t^2 + t + 1 = 0$ . We prove that the proper subspace  $S = \langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \rangle$  is invariant as follows:

$$(i) \quad \psi'_\lambda \sigma_1(e_1) = e_1 \in S.$$

$$(ii) \quad \psi'_\lambda \sigma_2(e_1) = e_1 \in S.$$

$$(iii) \quad \psi'_\lambda \sigma_3(e_1) = \begin{pmatrix} -t \\ t \\ \frac{tb_3}{b_1} \\ 0 \end{pmatrix} \in S.$$

$$(iv) \quad \psi'_\lambda \delta(e_1) = \begin{pmatrix} 1 \\ \frac{-t(1+t+t^2)}{(1+t)^2} \\ \frac{-t^2(1+t+t^2)}{(1+t)^2} + \frac{-t(1+t+t^2)}{b_1(1+t)}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \\ \frac{-t(1+t+t^2)}{(1+t)^2} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3).$$

Here, we have  $a = 1$ ,  $b = \frac{-d_3}{1+t}$ ,  $c = \frac{d_3}{1+t}$ , and  $\frac{cb_3}{b_1} = \frac{-t^2(1+t+t^2)}{(1+t)^2} + \frac{-t(1+t+t^2)}{b_1(1+t)}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)$ .

$$\text{Thus, } \frac{b_3}{b_1} = \frac{b_1 t + (1+t)(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)}{b_1}. \quad (5.1)$$

$$(v) \quad \psi'_\lambda \sigma_1(e_4) = -te_4 \in S.$$

$$(vi) \quad \psi'_\lambda \sigma_2(e_4) = \begin{pmatrix} 1 \\ \frac{-(1+t+t^2)}{1+t} \\ \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3).$$

Here, we have  $a = 1$ ,  $b = \frac{1}{1+t}$ ,  $c = \frac{-(1+t+t^2)}{1+t}$ , and  $\frac{cb_3}{b_1} = \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)}$ .

$$\text{Thus, } -(1+t+t^2)\frac{b_3}{b_1} = \frac{t(b_2+b_2t+b_1t)}{b_1}. \quad (5.2)$$

$$(vii) \quad \psi'_\lambda \sigma_3(e_4) = e_4 \in S.$$

$$(viii) \quad \psi'_\lambda \delta(e_4) = \begin{pmatrix} 0 \\ \frac{t}{1+t} \\ \frac{(-d_1(1+t)+d_2+b_1t)((\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)(1+t)+b_1 t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3).$$

Here, we have  $a = 0$ ,  $b = \frac{1}{1+t}$ ,  $c = \frac{t}{1+t}$ , and  $\frac{cb_3}{b_1} = \frac{(-d_1(1+t)+d_2+b_1t)((\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)(1+t)+b_1 t)}{b_1(1+t)}$ .

$$\text{Thus, } \frac{b_3}{b_1} = \frac{(-d_1(1+t)+d_2+b_1t)((\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)(1+t)+b_1 t)}{b_1 t}. \quad (5.3)$$

$$(ix) \quad \psi'_\lambda \sigma_1(e_2 + \frac{b_3}{b_1}e_3) = e_2 + \frac{b_3}{b_1}e_3 \in S.$$

$$(x) \quad \psi'_\lambda \sigma_2(e_2 + \frac{b_3}{b_1}e_3) = \begin{pmatrix} 0 \\ \frac{-t^2}{1+t} \\ \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)} + \frac{b_3}{b_1} \\ \frac{-t}{1+t} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3).$$

Here, we have  $a = 1$ ,  $b = \frac{-t}{1+t}$ ,  $c = \frac{-t^2}{1+t}$ , and  $\frac{cb_3}{b_1} = \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)} + \frac{b_3}{b_1}$ .

$$\text{Thus, } -(1+t+t^2)\frac{b_3}{b_1} = \frac{t(b_2+b_2t+b_1t)}{b_1}. \quad (5.4)$$

$$(xi) \quad \psi'_\lambda \sigma_3(e_2 + \frac{b_3}{b_1}e_3) = e_2 + \frac{b_3}{b_1}e_3 \in S.$$

$$(xii) \quad \psi'_\lambda \delta(e_2 + \frac{b_3}{b_1}e_3) =$$

$$\begin{pmatrix} 0 \\ 1 + \frac{d_2}{1+t} + \frac{b_3}{1+t} \\ \frac{d_2}{b_1}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \frac{b_1 t}{1+t}) + \frac{b_3}{b_1}(1 + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \frac{b_1 t}{1+t}) \\ \frac{-d_2}{1+t} - \frac{b_3}{1+t} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3).$$

Here, we have  $a = 0$ ,  $b = \frac{-d_2}{1+t} - \frac{b_3}{1+t}$ ,  $c = 1 + \frac{d_2}{1+t} + \frac{b_3}{1+t}$ , and  $c \frac{b_3}{b_1} = \frac{d_2}{b_1}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \frac{b_1 t}{1+t}) + \frac{b_3}{b_1}(1 + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \frac{b_1 t}{1+t})$ .

Thus,  $(1 + \frac{d_2}{1+t} + \frac{b_3}{1+t}) \frac{b_3}{b_1} = \frac{d_2}{b_1}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \frac{b_1 t}{1+t}) + \frac{b_3}{b_1}(1 + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \frac{b_1 t}{1+t})$ .  
(5.5)

By simple computations, we can verify that equations (5.1), (5.2), (5.3), and (5.5) are clearly satisfied without any assumption of the indeterminates whereas equation (5.4) is satisfied only if  $t^4 + t^3 + t^2 + t + 1 = 0$ .

**Lemma 3.** Any proper subspace  $S$  containing the vector  $e_i + ue_j + ve_k$ , where  $i, j, k \in \{1, 2, 3, 4\}$ , except possibly the subspace having the form  $\langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \rangle$ , is not invariant.

*Proof.* We consider all the subspaces containing the vector  $e_i + ue_j + ve_k$ , where  $i, j, k \in \{1, 2, 3, 4\}$  except possibly the subspace of the form  $\langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \rangle$ . We then assume, for contradiction, that each considered subspace is invariant. In each case, simple computations give a contradiction.

Thus, we have determined a necessary and sufficient condition for irreducibility.

**Theorem 1.** Assume all the indeterminates used in defining Perron representation of degree 4 are non zero complex numbers. Let  $d_3 = \frac{-t(1+t+t^2)}{1+t}$  and  $t \neq -1$ . The representation  $\psi'_\lambda : A(E_{4,1}) \rightarrow GL_4(\mathbb{C})$  is irreducible if and only if  $t^4 + t^3 + t^2 + t + 1 \neq 0$ .

In the following sections, we set  $n = 4$  and  $p = 1$  and we study the irreducibility of the reduced representation of  $\psi_\lambda : A(E_{5,1}) \rightarrow GL_8(\mathbb{C})$ . Indeed, we obtain a sufficient and necessary condition that guarantees the irreducibility of  $\psi'_\lambda : A(E_{5,1}) \rightarrow GL_5(\mathbb{C})$ .

## 6. Reducibility of $\psi_\lambda : A(E_{5,1}) \rightarrow GL_8(\mathbb{C})$

Having defined Perron's representation, we set  $n = 4$  and  $p = 1$  to get the following vectors.  $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ ,  $d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$ , and  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ .

After we specialize the indeterminates  $d_2$  and  $d_3$  to  $-(1+t+t^2)$  and  $-t(1+t)$  respectively, we get the following  $4 \times 4$  matrices:

$$A = \begin{pmatrix} \lambda_1 b_1 & \lambda_2 b_1 & \lambda_3 b_1 & \lambda_4 b_1 \\ \lambda_1 b_2 & \lambda_2 b_2 & \lambda_3 b_2 & \lambda_4 b_2 \\ \lambda_1 b_3 & \lambda_2 b_3 & \lambda_3 b_3 & \lambda_4 b_3 \\ \lambda_1 b_4 & \lambda_2 b_4 & \lambda_3 b_4 & \lambda_4 b_4 \end{pmatrix},$$

$$B = \begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 \\ b_4 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \lambda_1 d_1 & \lambda_2 d_1 & \lambda_3 d_1 & \lambda_4 d_1 \\ -(1+t+t^2)\lambda_1 & -(1+t+t^2)\lambda_2 & -(1+t+t^2)\lambda_3 & -(1+t+t^2)\lambda_4 \\ -t(1+t)\lambda_1 & -t(1+t)\lambda_2 & -t(1+t)\lambda_3 & -t(1+t)\lambda_4 \\ \lambda_1 d_4 & \lambda_2 d_4 & \lambda_3 d_4 & \lambda_4 d_4 \end{pmatrix},$$

and

$$D = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ -(1+t+t^2) & 0 & 0 & 0 \\ -t(1+t) & 0 & 0 & 0 \\ d_4 & 0 & 0 & 0 \end{pmatrix}.$$

Simple computations show that the parameters satisfy the following equations:

- $tb_2 = -td_1 - (1+t)(1+t+t^2) + t(1+t) = -td_1 - (1+t)(1+t^2)$
- $tb_3 = t(1+t+t^2) - t(1+t)^2 - d_4 = -t(2t^2 + 3t + 2) - d_4$
- $tb_4 = t^2(1+t) + (1+t)d_4$
- $tb_1 = (1+t)d_1 + 1 + 2t + t^2$
- $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4 = -(1+t+d_1)$

Having defined the  $4 \times 4$  matrices  $A$ ,  $B$ ,  $C$  and  $D$ , we obtain the multiparameter representation  $A(E_{5,1})$ . This representation is of degree 8. We specialize the parameters  $\lambda_1, \lambda_2, \lambda_3, \lambda_4, b_1, b_2, b_3, b_4, d_1, d_4, t$  to values in  $\mathbb{C} - \{0\}$ . We further assume that  $t \neq -1$ . The representation  $\psi_\lambda : A(E_{5,1}) \rightarrow GL_8(\mathbb{C})$  is defined as follows:

$$\psi_{\lambda}(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & -t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_{\lambda}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_{\lambda}(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & t & -t & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_{\lambda}(\sigma_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & t & -t \end{pmatrix},$$

and

$$\psi_\lambda(\delta) =$$

$$\begin{pmatrix} 1 + \lambda_1 b_1 & \lambda_2 b_1 & \lambda_3 b_1 & \lambda_4 b_1 & b_1 & 0 & 0 & 0 \\ \lambda_1 b_2 & 1 + \lambda_2 b_2 & \lambda_3 b_2 & \lambda_4 b_2 & b_2 & 0 & 0 & 0 \\ \lambda_1 b_3 & \lambda_2 b_3 & 1 + \lambda_3 b_3 & \lambda_4 b_3 & b_3 & 0 & 0 & 0 \\ \lambda_1 b_4 & \lambda_2 b_4 & \lambda_3 b_4 & 1 + \lambda_4 b_4 & b_4 & 0 & 0 & 0 \\ \lambda_1 d_1 & \lambda_2 d_1 & \lambda_3 d_1 & \lambda_4 d_1 & 1 + d_1 & 0 & 0 & 0 \\ k\lambda_1 & k\lambda_2 & k\lambda_3 & k\lambda_4 & k & 1 & 0 & 0 \\ -t(1+t)\lambda_1 & -t(1+t)\lambda_2 & -t(1+t)\lambda_3 & -t(1+t)\lambda_4 & -t(1+t) & 0 & 1 & 0 \\ \lambda_1 d_4 & \lambda_2 d_4 & \lambda_3 d_4 & \lambda_4 d_4 & d_4 & 0 & 0 & 1 \end{pmatrix},$$

where  $k = -(1 + t + t^2)$ .

The graph  $E_{5,1}$  has 5 vertices  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  and  $\delta$ . Since  $p = 1$ , it follows that the vertex  $\delta$  has a common edge with  $\sigma_p = \sigma_1$ . Therefore, the following relations are satisfied.

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \quad (6.1)$$

$$\sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3 \quad (6.2)$$

$$\sigma_3 \sigma_4 \sigma_3 = \sigma_4 \sigma_3 \sigma_4 \quad (6.3)$$

$$\sigma_1 \sigma_3 = \sigma_3 \sigma_1 \quad (6.4)$$

$$\sigma_1 \sigma_4 = \sigma_4 \sigma_1 \quad (6.5)$$

$$\sigma_2 \sigma_4 = \sigma_4 \sigma_2 \quad (6.6)$$

$$\sigma_2 \delta = \delta \sigma_2 \quad (6.7)$$

$$\sigma_3 \delta = \delta \sigma_3 \quad (6.8)$$

$$\sigma_4 \delta = \delta \sigma_4 \quad (6.9)$$

$$\sigma_1 \delta \sigma_1 = \delta \sigma_1 \delta \quad (6.10)$$

We note that relations (6.1), (6.2), (6.3), (6.4), (6.5) and (6.6) are actually Artin's braid relation of the classical braid group,  $B_5$  having  $\sigma_1, \sigma_2, \sigma_3$ , and  $\sigma_4$  as standard generators. This assures that a representation of  $A(E_{5,1})$  yields a representation of  $B_5$ . For more details, see [1] and [4].

**Lemma 4.** *The representation  $\psi_\lambda : A(E_{5,1}) \rightarrow GL_8(\mathbb{C})$  is reducible.*

*Proof.* For simplicity, we write  $\sigma_i$  instead of  $\psi_\lambda(\sigma_i)$ . The subspace  $S = \left\langle e_1 + \frac{b_2}{b_1} e_2 + \frac{b_3}{b_1} e_3 + \frac{b_4}{b_1} e_4, e_5, e_6, e_7, e_8 \right\rangle$  is an invariant subspace of dimension 5.

### 7. On the Irreducibility of $\psi'_\lambda : A(E_{5,1}) \rightarrow GL_5(\mathbb{C})$

We consider the representation  $\psi_\lambda : A(E_{5,1}) \rightarrow GL_8(\mathbb{C})$  restricted to the basis  $e_1, e_2, e_3, e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + \frac{b_4}{b_1}e_4, e_5, e_6, e_7$ , and  $e_8$  to get the subrepresentation  $\psi'_\lambda : A(E_{5,1}) \rightarrow GL_5(\mathbb{C})$  which is the representation restricted to the sub-basis  $e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + \frac{b_4}{b_1}e_4, e_5, e_6, e_7$ . This representation is defined as follows:

$$\psi'_\lambda(\sigma_1) = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_2) = \begin{pmatrix} 1 & 0 & \frac{tb_2}{b_1} & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & \frac{tb_3}{b_1} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{tb_4}{b_1} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & -t \end{pmatrix},$$

and

$$\psi'_\lambda(\delta) =$$

$$\begin{pmatrix} 1+r & \frac{d_1}{b_1}r & \frac{-(1+t+t^2)}{b_1}r & \frac{-t(1+t)}{b_1}r & \frac{d_4}{b_1}r \\ b_1 & 1+d_1 & -(1+t+t^2) & -t(1+t) & d_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $r = \sum_{i=1}^4 \lambda_i b_i$ .

We then diagonalize the matrix corresponding to  $\psi'_\lambda(\sigma_1)$  by an invertible matrix, say  $T$ , and conjugate the matrices of  $\psi'_\lambda(\sigma_2)$ ,  $\psi'_\lambda(\sigma_3)$ ,  $\psi'_\lambda(\sigma_4)$  and  $\psi'_\lambda(\delta)$  by the same matrix  $T$ . The invertible matrix  $T$  is given by

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & -1-t \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In fact, a computation shows that

$$T^{-1}\psi'_\lambda(\sigma_1)T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -t \end{pmatrix}.$$

After conjugation, we get

$$T^{-1}\psi'_\lambda(\sigma_2)T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & \frac{-t^2}{1+t} & 0 & \frac{-(1+t+t^2)}{1+t} \\ 0 & 0 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} & 1 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ 0 & 0 & \frac{-t}{1+t} & 0 & \frac{1}{1+t} \end{pmatrix},$$

$$T^{-1}\psi'_\lambda(\sigma_3)T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 \\ 0 & t & 1 & 0 & 0 \\ 0 & \frac{tb_3}{b_1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T^{-1}\psi'_\lambda(\sigma_4)T = \begin{pmatrix} -t & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{tb_4}{b_1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$T^{-1}\psi'_\lambda(\delta)T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{d_4}{1+t} & -t & 1 + \frac{-(1+t+t^2)}{1+t} & \frac{b_1}{1+t} & \frac{t}{1+t} \\ \frac{d_4}{b_1}w & \frac{-t(1+t)}{b_1}w & \frac{-(1+t+t^2)}{b_1}w & 1+w & \frac{(-d_1(1+t)-(1+t+t^2)+b_1t)}{b_1}w \\ \frac{-d_4}{1+t} & t & \frac{1+t+t^2}{1+t} & \frac{-b_1}{1+t} & \frac{1}{1+t} \end{pmatrix},$$

where  $w = \sum_{i=1}^4 \lambda_i b_i + \frac{b_1 t}{1+t}$ .

The entries of the matrices  $T^{-1}\psi'_\lambda(\sigma_2)T, T^{-1}\psi'_\lambda(\sigma_3)T, T^{-1}\psi'_\lambda(\sigma_4)T$  and  $T^{-1}\psi'_\lambda(\delta)T$  are well-defined since we assume in our work that  $t \neq -1$ . For simplicity, we denote  $T^{-1}\psi'_\lambda(\sigma_1)T$  by  $\psi'_\lambda(\sigma_1)$ ,  $T^{-1}\psi'_\lambda(\sigma_2)T$  by  $\psi'_\lambda(\sigma_2)$ ,  $T^{-1}\psi'_\lambda(\sigma_3)T$  by  $\psi'_\lambda(\sigma_3)$ ,  $T^{-1}\psi'_\lambda(\sigma_4)T$  by  $\psi'_\lambda(\sigma_4)$ , and  $T^{-1}\psi'_\lambda(\delta)T$  by  $\psi'_\lambda(\delta)$ .

We now prove some lemmas and propositions to determine a sufficient and necessary condition for irreducibility of  $\psi'_\lambda : A(E_{5,1}) \rightarrow GL_5(\mathbb{C})$ .

**Lemma 5.** *Except possibly the subspaces having the forms  $\langle e_1, e_3, e_5, e_2 + ue_4 \rangle$  and  $\langle e_2, e_3, e_5, e_1 + ue_4 \rangle$ , where  $u \in \mathbb{C}^*$ , every proper subspace is not invariant.*

*Proof.* We assume, for contradiction, that every subspace, except those having the forms  $\langle e_1, e_3, e_5, e_2 + ue_4 \rangle$  and  $\langle e_2, e_3, e_5, e_1 + ue_4 \rangle$ , is invariant.

We then study each possible form. In each case, simple computations give a contradiction.

**Lemma 6.** *If  $t^3 \neq -1$ , then the subspaces  $\langle e_1, e_3, e_5, e_2 + ue_4 \rangle$  and  $\langle e_2, e_3, e_5, e_1 + ue_4 \rangle$  are not invariant.*

*Proof.* First, we assume, for contradiction, that  $S = \langle e_1, e_3, e_5, e_2 + ue_4 \rangle$  is invariant.

$$\bullet \psi'_\lambda \sigma_4(e_1) = \begin{pmatrix} -t \\ t \\ 0 \\ \frac{tb_4}{b_1} \\ 0 \end{pmatrix} = ae_1 + be_3 + ce_5 + d(e_2 + ue_4) \text{ which implies that } u = \frac{b_4}{b_1}.$$

(7.1)

$$\bullet \psi'_\lambda \sigma_2(e_3) = \begin{pmatrix} 0 \\ 1 \\ \frac{-t^2}{1+t} \\ \frac{t(b_1t+b_2+b_2t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} = ae_1 + be_3 + ce_5 + d(e_2 + ue_4) \text{ which implies that}$$

$$u = \frac{t(b_1t+b_2+b_2t)}{b_1(1+t)}.$$

(7.2)

$$\bullet \psi'_\lambda \sigma_3(e_2 + ue_4) = \begin{pmatrix} 1 \\ -t \\ t \\ u + \frac{tb_3}{b_1} \\ 0 \end{pmatrix} = ae_1 + be_3 + ce_5 + d(e_2 + ue_4) \text{ which implies that}$$

$$u = \frac{-tb_3}{b_1(1+t)}.$$

(7.3)

Since equations (7.1) and (7.3) are equal, we have  $(1+t+t^2)d_4 = -t^2(1+t+t^2)$ . Thus,  $d_4 = -t^2$ .

Moreover, equations (7.2) and (7.3) are equal. This implies that  $d_4 = -(1+t^2)^2 + t^2$ . By substituting  $d_4 = -t^2$ , we get  $t^4 + t^3 + t + 1 = (t+1)(t^3+1) = 0$ , a contradiction.

Now, we assume, for contradiction, that  $S = \langle e_2, e_3, e_5, e_1 + ue_4 \rangle$  is invariant.

$$\bullet \psi'_\lambda \sigma_2(e_3) = \begin{pmatrix} 0 \\ 1 \\ \frac{-t^2}{1+t} \\ \frac{t(b_1t+b_2+b_2t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4).$$

This implies that  $t(b_1t + b_2 + b_2t) = 0$ .

Simple computations give  $-(1+t+t^2)^2 + t(1+t)^2 + t^2 = 0$ . Thus,  $t^4 + t^3 + t + 1 = 0$ , a contradiction.

We now determine conditions under which one of the subspaces mentioned in Lemma 7 is invariant. But first we write down the following lemma.

**Lemma 7.** *The proper subspaces  $S_1 = \langle e_1, e_3, e_5, e_2 + ue_4 \rangle$  and  $S_2 = \langle e_2, e_3, e_5, e_1 + ue_4 \rangle$  cannot be both invariant.*

*Proof.* Assume that  $S_1$  is invariant. This implies that  $\psi'_\lambda \sigma_2(e_3)$  and  $\psi'_\lambda \sigma_3(e_2 + ue_4) \in S_1$ . Simple computations give  $b_1t + b_2 + b_2t = -b_3 \neq 0$ .

Assume, for contradiction, that  $S_2$  is invariant. This implies that  $\psi'_\lambda \sigma_2(e_3) \in S_2$ . Simple computations give  $b_1t + b_2 + b_2t = 0$ , a contradiction.

**Lemma 8.** *If  $t^3 = -1$ , then the subspace  $S = \langle e_2, e_3, e_5, e_1 + ue_4 \rangle$  is invariant.*

*Proof.*

$$\bullet \psi'_\lambda \sigma_1(e_2) = \psi'_\lambda \sigma_2(e_2) = \psi'_\lambda \sigma_4(e_2) = e_2 \in S.$$

$$\bullet \psi'_\lambda \sigma_3(e_2) = \begin{pmatrix} 1 \\ -t \\ t \\ \frac{tb_3}{b_1} \\ 0 \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$$

if  $u = \frac{tb_3}{b_1}$ . (7.4)

$$\bullet \psi'_\lambda \delta(e_2) = \begin{pmatrix} 0 \\ 1 \\ \frac{d_3}{1+t} \\ \frac{d_3 t}{1+t} + \frac{d_3}{b_1} (\sum_{i=1}^4 \lambda_i b_i) \\ \frac{-d_3}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$$

if  $\frac{t}{1+t} = \frac{-\sum_{i=1}^4 \lambda_i b_i}{b_1}$ . (7.5)

$$\bullet \psi'_\lambda \sigma_1(e_3) = \psi'_\lambda \sigma_3(e_3) = \psi'_\lambda \sigma_4(e_3) = e_3 \in S.$$

$$\bullet \psi'_\lambda \sigma_2(e_3) = \begin{pmatrix} 0 \\ 1 \\ \frac{-t^2}{1+t} \\ \frac{t(b_1 t + b_2 + b_2 t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$$

if  $t(b_1 t + b_2 + b_2 t) = 0$ . (7.6)

$$\bullet \psi'_\lambda \delta(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 + \frac{-(1+t+t^2)}{1+t} \\ \frac{-t^2(1+t+t^2)}{1+t} + \frac{-(1+t+t^2)}{\frac{b_1}{1+t+t^2}} (\sum_{i=1}^4 \lambda_i b_i) \\ \frac{1+t+t^2}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$$

if  $\sum_{i=1}^4 \lambda_i b_i + \frac{b_1 t}{1+t} = 0$ . (7.7)

$$\bullet \psi'_\lambda \sigma_1(e_5) = -te_5 \in S.$$

$$\bullet \psi'_\lambda \sigma_3(e_5) = \psi'_\lambda \sigma_4(e_5) = e_5 \in S.$$

$$\bullet \psi'_\lambda \sigma_2(e_5) = \begin{pmatrix} 0 \\ 1 \\ \frac{-(1+t+t^2)}{1+t} \\ \frac{t(b_1t+b_2+b_2t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$$

$$\text{if } t(b_1t + b_2 + b_2t) = 0. \quad (7.8)$$

$$\bullet \psi'_\lambda \delta(e_5) = \begin{pmatrix} 0 \\ 0 \\ \frac{t}{1+t} \\ \frac{(-d_1(1+t)-t(1+t+t^2)+b_2t)(\sum_{i=1}^4 \lambda_i b_i(1+t)+b_1t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$$

$$\text{if } (-d_1(1+t) + d_2 + b_2t)(\sum_{i=1}^4 \lambda_i b_i(1+t) + b_1t) = 0. \quad (7.9)$$

$$\bullet \psi'_\lambda \sigma_1(e_1 + ue_4) = \psi'_\lambda \sigma_2(e_1 + ue_4) = \psi'_\lambda \sigma_3(e_1 + ue_4) = e_1 + ue_4 \in S.$$

$$\bullet \psi'_\lambda \sigma_4(e_1 + ue_4) = \begin{pmatrix} -t \\ t \\ 0 \\ u + \frac{tb_4}{b_1} \\ 0 \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$$

$$\text{if } u = \frac{-tb_4}{b_1(1+t)}. \quad (7.10)$$

$$\bullet \psi'_\lambda \delta(e_1 + ue_4) = \begin{pmatrix} 1 \\ 0 \\ \frac{d_4}{1+t} + \frac{b_1u}{1+t} \\ \frac{d_4}{b_1}(\sum_{i=1}^4 \lambda_i b_i + \frac{b_1t}{1+t}) + u(1 + \sum_{i=1}^4 \lambda_i b_i + \frac{b_1t}{1+t}) \\ \frac{-d_4}{1+t} - \frac{b_1u}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$$

$$\text{if } (\sum_{i=1}^4 \lambda_i b_i + \frac{b_1t}{1+t})(\frac{d_4}{b_1} + u) = 0. \quad (7.11)$$

Using the relations, we prove that equations (7.5), (7.7), (7.9), and (7.11) are clearly satisfied.

Also, we verify that equations (7.4), (7.6), (7.8) and (7.10) are satisfied if  $-t(1+t)^2 = -(1+t+t^2)(1+t^2) + t^2$  which implies that  $t^3 = -1$ .

Thus, we have determined a necessary and sufficient condition for irreducibility.

**Theorem 2.** *Assume all the indeterminates used in defining Perron representation of degree 5 are non zero complex numbers. Let  $d_2 = -(1+t+t^2)$ ,  $d_3 = -t(1+t)$ , and  $t \neq -1$ . The representation  $\psi'_\lambda : A(E_{5,1}) \rightarrow GL_5(\mathbb{C})$  is irreducible if and only if  $t^3 \neq -1$ .*

**Remark 1.** • For  $n=2$  and for  $t \neq -1$ , we proved that a complex specialization of the representation  $\psi'_\lambda : A(E_{3,1}) \rightarrow Gl_3(\mathbb{C})$  is irreducible if and only if  $t^2 \neq -1$  which is equivalent to  $t^3 + t^2 + t + 1 \neq 0$ .

- For  $n=3$  and for  $t \neq -1$ , we have proved that a complex specialization of the representation  $\psi'_\lambda : A(E_{4,1}) \rightarrow Gl_4(\mathbb{C})$  is irreducible if and only if  $t^4 + t^3 + t^2 + t + 1 \neq 0$ .
- For  $n=4$  and for  $t \neq -1$ , we have proved that a complex specialization of the representation  $\psi'_\lambda : A(E_{5,1}) \rightarrow Gl_5(\mathbb{C})$  is irreducible if and only if  $t^3 \neq -1$  which is equivalent to  $t^5 + t^4 + t^3 + t^2 + t + 1 \neq 0$ .

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